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Existence of solutions of a class of fractional partial integrodifferential equations

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Abstract. In this paper we study the existence of solutions for the nonlinear Sobolev type fractional partial integrodifferential equations with Neumann boundary condition. Under suitable assumptions the results are established by using the Leray-Schauder fixed point theorem.

1. Introduction

There has been an increasing interest in recent years shown by many researchers from various fields of science and engineering to study fractional order partial differential equations [18, 21, 24, 25, 28, 29]. These equations are found to be an effective tool to describe certain physical phenomena, such as diffusion processes [19] and viscoelasticity theories [20]. Several authors have discussed the existence of solutions for different types of fractional partial differential equations [1–3, 15, 27].

Brill [14] and Showalter [30] investigated the problem of existence of solutions of for semilinear Sobolev-type equations in Banach spaces. The Sobolev-type semilinear differential equation serves as an abstract formulation of partial differential equations which arise in various applications such as in the flow of fluid through fissured rocks [13], thermodynamics and shear in second order fluids. Lightbourne and Rankin [26] discussed the Cauchy problem for a partial functional differential equation of Sobolev-type. Balachandran et al. [12] established the existence results for Sobolev-type semilinear integrodifferential equations whereas Balachandran and Uchiyama [10] studied the same problem for nonlinear integrodifferential equation of Sobolev-type with nonlocal condition in Banach spaces. Several authors have studied the nonlocal Cauchy problem for Sobolev-type differential equations in Banach spaces [7–9, 11].

Balachandran and Kiruthika [6] investigated the existence problem for abstract fractional integrodifferential equations of Sobolev-type. Existence

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of solutions for abstract fractional integrodifferential equations of Sobolev-type with deviating arguments are studied in [22]. The existence results for fractional order Sobolev-type partial differential equations was discussed in [5]. In this paper, we study the existence problem for fractional partial integrodifferential equations of Sobolev-type with Neumann boundary condition by suitably adopting the technique of [27].

2. Preliminaries

In this section, we introduce some notations and basic facts of fractional calculus. Let $\Omega \subset \mathbb{R}$ and $C(J, \mathbb{R})$ is the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$. Let $\Gamma(\cdot)$ denote the gamma function. For any positive number $0 < \alpha < 1$, the Riemann Liouville derivative and Caputo derivative are defined as follows.

Definition 1 ([23])**.** The Riemann-Liouville partial fractional integral operator of order $0 < \alpha < 1$ with respect to t of a function $z(x, t)$ is defined by

$$I^{\alpha} z(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(x,s)\, \mathrm{d}s.$$

Definition 2 ([23])**.** The Riemann-Liouville partial fractional derivative of order $0 < \alpha < 1$ of a function $z(x, t)$ with respect to t is defined by

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{z(x,s)}{(t-s)^{\alpha}}\, \mathrm{d}s.$$

Definition 3 ([23])**.** The Caputo partial fractional derivative of order $0 < \alpha < 1$ with respect to t of a function $z(x, t)$ is defined as

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{\partial z(x,s)}{\partial s}\, \mathrm{d}s.$$

A significant research development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives is contained in [4, 5, 27]. The Riemann-Liouville and Caputo fractional derivatives are linked by the relationship

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}} z(x,t) - \frac{z(x,0)}{\Gamma(1-\alpha) t^{\alpha}}.$$

Further the property

$$I^{\alpha} \frac{{}^{C}\partial^{\alpha} z(x,t)}{\partial t^{\alpha}} = z(x,t) - z(x,0) \tag{1}$$

is useful in proving the existence results.

Theorem 1 (Leray-Schauder fixed point theorem, see [17]). *If U is a closed bounded convex subset of a Banach space X and $P : U \to U$ is completely continuous, then P has at least a fixed point in U.*

Theorem 2 (Green's Identity, see [16]). *Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Then, for any $u, v \in C^2(\Omega)$,*

$$\int_\Omega v\Delta u \, \mathrm{d}x = \int_{\partial\Omega} v\frac{\partial u}{\partial n} \, \mathrm{d}s - \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x,$$

where n is the outward unit normal to the boundary $\partial\Omega$ and $\mathrm{d}s$ is the element of arc length. For the special case $v = 1$,

$$\int_\Omega \Delta u \, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s. \tag{2}$$

This is called Green's first identity.

In this paper, we consider Sobolev-type fractional partial integrodifferential equations for $t \in J$ of the form

$$\begin{aligned} &\frac{{}^C\partial^\alpha}{\partial t^\alpha}[u(x,t) - \Delta u(x,t)] \\ &= \Delta u(x,t) + f\left(t, x, u(x,t), \int_0^t k(t,s,x,u(x,s))\mathrm{d}s\right), \end{aligned} \tag{3}$$

where $0 < \alpha < 1$, Ω is a bounded subset of $\mathbb{R}$ with smooth boundary $\partial\Omega$, $J = [0, T]$ and $f : J \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : J \times J \times \Omega \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions. The initial and Neumann boundary conditions are given by

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{4}$$

$$\frac{\partial u(x,t)}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times J, \tag{5}$$

where $u_0(x) \in C^2(\Omega)$. In order to establish the main result, we assume the following conditions:

(H$_1$) $f(t, x, u_1, u_2)$ is continuous with respect to u_1, u_2, Lebesgue measurable with respect to t and satisfies

$$\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f(t, x, u_1, u_2) \, \mathrm{d}x \le f\left(t, \frac{\int_\Omega u_1(x,t) \, \mathrm{d}x}{\mathrm{vol}(\Omega)}, \frac{\int_\Omega u_2(x,t) \, \mathrm{d}x}{\mathrm{vol}(\Omega)}\right).$$

(H$_2$) There exists an integrable function $m_1 : J \to [0, \infty)$ and a constant $L_1 > 0$ such that

$$\|f(t, x, u_1, u_2)\| \le m_1(t)(\|u_1\| + \|u_2\|),$$

$$\int_0^T (T - s)^{\alpha-1} m_1(s)\mathrm{d}s \le L_1 \text{ for some } \alpha > 0.$$

(H_3) $k(t,s,x,u)$ is continuous with respect to u, Lebesgue measurable with respect to t and satisfies

$$\frac{1}{\operatorname{vol}(\Omega)}\int_\Omega k(t,s,x,u)\,\mathrm{d}x \le k\left(t,s,\frac{\int_\Omega u(x,t)\,\mathrm{d}x}{\operatorname{vol}(\Omega)}\right).$$

(H_4) There exists an integrable function $m_2(t,s): J\times J\to[0,\infty)$ and a constant $L_2>0$ such that

$$\|k(t,s,x,u)\|\le m_2(t,s)\|u\|,$$

$$\int_0^T (T-s)^{\alpha-1}m_1(s)\int_0^s m_2(s,\tau)d\tau\mathrm{d}s \le L_2 \text{ for some } \alpha>0,$$

$$L_1+L_2\ne\Gamma(\alpha).$$

It is easy to show that by using (1), the fractional partial integrodifferential equation (3) with (4) is equivalent to the equation

$$\begin{aligned} u(x,t) &= u_0(x)-u_0''(x)+\Delta u(x,t)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta u(x,s)\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x,u(x,s),v(x,s))\,\mathrm{d}s, \end{aligned}\tag{6}$$

where

$$v(x,s)=\int_0^s k(s,\tau,x,u(x,\tau))\mathrm{d}\tau.$$

3. EXISTENCE RESULT

In this section, we establish the main result of this paper. For this, we define the functions $U(t)$ and $V(t)$ as

$$U(t)=\frac{1}{\operatorname{vol}(\Omega)}\int_\Omega u(x,t)\,\mathrm{d}x$$

$$V(t)=\frac{1}{\operatorname{vol}(\Omega)}\int_\Omega v(x,t)\,\mathrm{d}x.$$

Theorem 3. *Suppose that* (H_1)–(H_4) *hold. Then there exists at least one solution for the initial value problem* (3) *on* J.

Proof. We have to prove that the initial value problem (3) has a solution if and only if the equation

$$U(t)=U(0)-U''(0)+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}f(s,U(s),V(s))\,\mathrm{d}s, \tag{7}$$

where

$$U''(0)=\frac{1}{\mathrm{vol}(\Omega)}\int\limits_\Omega \Delta u(x,0)\mathrm{d}x=\frac{1}{\mathrm{vol}(\Omega)}\int\limits_\Omega u_0''(x)\mathrm{d}x$$

has a solution.

Assume $u(x,t)$ to be a solution of (3). Then it follows that $u(x,t)$ is a solution of (6). Now integrating both sides of equation (6) with respect to $x\in\Omega$, we get

$$\begin{aligned}\int\limits_\Omega u(x,t)\ \mathrm{d}x=&\int\limits_\Omega u_0(x)\ \mathrm{d}x-\int\limits_\Omega u_0''(x)\ \mathrm{d}x+\int\limits_\Omega \Delta u(x,t)\ \mathrm{d}x\\ &+\frac{1}{\Gamma(\alpha)}\int\limits_\Omega\int_0^t (t-s)^{\alpha-1}\Delta u(x,s)\ \mathrm{d}s\ \mathrm{d}x\\ &+\frac{1}{\Gamma(\alpha)}\int\limits_\Omega\int_0^t (t-s)^{\alpha-1}f(s,u(x,s),v(x,s))\ \mathrm{d}s\ \mathrm{d}x\end{aligned}$$

Using the Green identity (2) and the Neumann boundary condition (5) we get

$$\int\limits_\Omega \Delta u(x,t)\mathrm{d}x=0.$$

By the assumptions (H_1) and (H_3), we have

$$U(t)\le U(0)-U''(0)+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}f(s,U(s),V(s))\ \mathrm{d}s. \tag{8}$$

Choose $b\ge\dfrac{\|U(0)\|(L_1+L_2)}{\Gamma(\alpha)-(L_1+L_2)}$ and let

$$K=\{U:U\in C(J,\mathbb{R}),\|\ U(t)-U(0)\ \|\le b\}.$$

Define the nonlinear operator $F:C(J,\mathbb{R})\to C(J,\mathbb{R})$ as

$$FU(t)=U(0)-U''(0)+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}f(s,U(s),V(s))\ \mathrm{d}s. \tag{9}$$

Clearly $U(0)\in K$ and so K is nonempty. Further K is a closed bounded convex set. Next we have to prove that the operator F maps K into itself.

Note that

$$\| FU(t) - FU(0) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, U(s), V(s))\| \, \mathrm{d}s.$$

Then by using (H_2) and (H_4), we get

$$\begin{aligned}
&\| FU(t) - FU(0) \| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t m_1(s)(t-s)^{\alpha-1}[\|U(s)\| + \|V(s)\|] \, \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} (\|U(0)\| + b) \int_0^T m_1(s)(T-s)^{\alpha-1}[1 + \int_0^s m_2(s,\tau)\mathrm{d}\tau]\mathrm{d}s \\
&\leq \frac{L_1 + L_2}{\Gamma(\alpha)} (\|U(0)\| + b) \\
&\leq b.
\end{aligned}$$

Therefore F maps K into itself. Using similar argument as in [5, 27], it is easy to show that F is completely continuous. Then by the Leray-Schauder fixed point theorem, F has a fixed point in K, which is a solution of the equation (3). □

Remark. Consider the Sobolev-type fractional partial integrodifferential equation with kernel of the form

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}[u(x,t) - \Delta u(x,t)] = \Delta u(x,t) + \int_0^t h(t-s)\Delta u(x,s)\mathrm{d}s + f(t,x,u(x,t)), \tag{10}$$

where $f : J \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are nonlinear continuous functions. Existence of solutions of equation (10) with Neumann boundary condition (5) can be established by using the same method as in Theorem 3.

4. ABSTRACT FRACTIONAL SOBOLEV EQUATION

The Sobolev-type fractional partial integrodifferential equation (3) with (4) can be discussed in abstract setting as in [6, 14, 26] and it can be written for $t \in J$ as

$$\begin{aligned}
&\frac{{}^C\partial^\alpha}{\partial t^\alpha}[Bu(x,t)] = Au(x,t) + f\left(t, x, u(x,t), \int_0^t k(t,s,x,u(x,t)\mathrm{d}s\right), \\
&u(x,0) = u_0(x),
\end{aligned} \tag{11}$$

where $B = I - A$ and $A = \Delta$. Suppressing the phase variable and letting X a general Banach space, (11) can be written as abstract fractional differential equation of the form

$$(12)\qquad \begin{aligned} {}^{C}D^{\alpha}[Bu(t)] &= Au(t) + f\left(t, u(t), \int_0^t k(t,s,u)\mathrm{d}s\right), \quad t \in J, \\ u(0) &= u_0, \end{aligned}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative with $0 < \alpha < 1$. The operators A and B are linear with domains contained in X and ranges contained in a Banach Space Y and the operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(C1) A and B are closed linear operators,
(C2) $D(B) \subset D(A)$ and B is bijective,
(C3) $B^{-1} : Y \to D(B)$ is compact,
(C4) $B^{-1}A : X \to D(B)$ is continuous.

The nonlinear operators $f : J \times X \times X \to Y$ and $k : J \times J \times X \to X$ are continuous. It is easy to prove that the equation (12) is equivalent to the integral equation

$$(13)\qquad \begin{aligned} u(t) = u_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B^{-1} Au(s)\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B^{-1} f\left(s, u(s), \int_0^s k(s,\tau,u)\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned}$$

By using the resolvent operators, the solvability problem for the equation (12) has been discussed in [6, 22].

5. EXAMPLE

Consider the Sobolev-type fractional partial integrodifferential problem of the form

$$(14)\qquad \begin{aligned} \frac{\partial^{\frac{3}{4}}}{\partial t^{\frac{3}{4}}}\left[u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x)\right] &= \frac{\partial^2}{\partial x^2}u(t,x) + b(t)\sin u(t,x) \\ &\quad + \int_0^t a(t-s)e^{-u(s,x)}\mathrm{d}s, \quad t > 0, \end{aligned}$$

$$(15)\qquad u(x,0) = u_0(x), \quad x \in \Omega,$$

$$(16)\qquad \frac{\partial u(x,t)}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times J,$$

where $J = [0,T], \Omega = [0,\pi]$, $u_0 \in C^2(\Omega)$ and $a, b \in C(J)$. Here the function $f : J \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ and $k : J^2 \times \Omega \times \mathbb{R} \to \mathbb{R}$ are given as

$$f(t,x,u(t,x),Ku(t,x)) = b(t)\sin u(t,x) + Ku(t,x),$$

$$Ku(t,x) = \int_0^t k(t,s,x,u(s,x))\mathrm{d}s = \int_0^t a(t-s)e^{-u(s,x)}\mathrm{d}s.$$

By appropriately choosing the functions $a(t), b(t)$ [6], one can impose the conditions $(H_1)-(H_4)$ of Theorem 3 on the functions f, k, and hence there is a function $u \in C(J,\mathbb{R})$ which satisfies the equations (14)–(16) on J.

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